Probabilistic tools in continuous combinatorics

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- If $V(G)$ is a Polish space, can define the Baire-measurable chromatic number $\chi_{\mathrm{BM}}(G)$ of G. ("Borel mod meager.")

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If X is a standard Borel space, then the Borel σ-algebra on X is generated by a zero-dimensional Polish topology.

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The free part: $\mathrm{Free}(2^\Gamma) := \{x \in 2^\Gamma : \gamma \cdot x \neq x \text{ for all } \gamma \neq \mathbf{1}_\Gamma\}$.

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Theorem (Kechris–Solecki–Todorcevic '91 + Marks '16)

Let $\mathbb{F}_n$ be the free group on n generators. Then

$$\chi_{\mathrm{B}}(\mathrm{Free}(2^{\mathbb{F}_n})) = \chi_{\mathrm{c}}(\mathrm{Free}(2^{\mathbb{F}_n})) = 2n+1.$$

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If $d \geqslant 2$, then $\chi_B(\mathrm{Free}(2^{\mathbb{Z}^d})) = 3$, but $\chi_c(\mathrm{Free}(2^{\mathbb{Z}^d})) = 4$.

General setting: constraint satisfaction problems

Fix a set X and a positive integer k.

A constraint is a set B of functions $D \to k$, where D is a finite subset of X called the domain of B. Write $\mathrm{dom}(B) := D$.

A function $f\colon X \to k$ violates B if $f|_D \in B$. Otherwise, f satisfies B.

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A constraint satisfaction problem (a CSP for short) is a set $\mathscr{B}$ of constraints. To indicate that $\mathscr{B}$ is a CSP, we write $\mathscr{B}\colon X \to^{?} k$.

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Example: In the graph coloring problem, $X = V(G)$ and there is one constraint per edge.

Probabilistic criterion for CSPs

Let $\mathscr{B}\colon X \to^{?} k$ be a CSP. The probability of a constraint $B \in \mathscr{B}$ is

$$\mathbb{P}[B] := \frac{|B|}{k^{|\mathrm{dom}(B)|}} = \text{prob. } B \text{ is violated by a random coloring.}$$

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Theorem (A.B.)

Let $\mathscr{B} : X \to^{?} k$ be a continuous CSP on a zero-dim. Polish space X.

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A sinkless orientation of G is a solution to a CSP

$$\mathscr{B}_{\text{sinkless}} = \{B_x\}_{x \in V(G)} : E(G) \to^{?} 2.$$

Here B_x is the constraint with domain $\{e \in E(G) : e \text{ is incident to } x\}$ saying that at least one edge must be leaving x.

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Therefore, $p(\mathscr{B}) \cdot \text{vdeg}(\mathscr{B})^{\text{ord}(\mathscr{B})} = 1$. But:

Theorem (Thornton)

For any $d \in \mathbb{N}$, there exists a d-regular Borel graph G without a Borel sinkless orientation.

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In terms of $p(\mathscr{B})$ and $d(\mathscr{B})$: Brandt–Grunau–Rozhoň + A.B.:

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$$p(\mathscr{B}) \cdot 2^{d(\mathscr{B})} < 1 \quad \Longrightarrow \quad \text{continuous solution.}$$

In practice, often use the bound $d(\mathscr{B}) \leqslant \text{ord}(\mathscr{B})(\text{vdeg}(\mathscr{B}) - 1)$, making the above theorem more widely applicable (especially if $\text{ord}(\mathscr{B})$ is small).

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For $B \in \mathscr{B}_x$ and a color α, write

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Key observation: For each $x \in X$, there is a color α such that $\mathbb{P}[B \mid x \mapsto \alpha] \leqslant \mathrm{vdeg} \cdot \mathbb{P}[B]$ for all $B \in \mathscr{B}_x$.

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Set vdeg := vdeg($\mathscr{B}$). For $x \in X$, let $\mathscr{B}_x := \{B \in \mathscr{B} : x \in \mathrm{dom}(B)\}$.

For $B \in \mathscr{B}_x$ and a color α, write

$$\mathbb{P}[B \,|\, x \mapsto \alpha] := \frac{|\{\varphi \in B : \varphi(x) = \alpha\}|}{k^{|\mathrm{dom}(B)|-1}}.$$

Key observation: For each $x \in X$, there is a color α such that $\mathbb{P}[B \,|\, x \mapsto \alpha] \leqslant \mathrm{vdeg} \cdot \mathbb{P}[B]$ for all $B \in \mathscr{B}_x$.

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We are done since $|\mathscr{B}_x| \leqslant \mathrm{vdeg}$. ∎

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Let $\mathscr{B} \colon X \to^{?} k$ be a continuous CSP on a zero-dim. Polish space X.

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In general: standard methods allow using greedy algorithms to construct continuous colorings.

Application: Seward–Tucker-Drob theorem

Γ a countably infinite group.

A subshift is a closed shift-invariant subset of 2^Γ.

A subshift X is free if $X \subseteq \mathrm{Free}(2^\Gamma)$.

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Which groups Γ admit a nonempty free subshift X?

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If $\Gamma \curvearrowright X$ is a free Borel action on a standard Borel space X, then there is an equivariant Borel map $\pi \colon X \to Y$, where $Y \subseteq \mathrm{Free}(2^\Gamma)$ is a free subshift.

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Given an action $\Gamma \curvearrowright X$ and a coloring $f\colon X \to k$, say that a pattern $p\colon D \to k$ occurs in f if there is $x \in X$ such that

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Otherwise, f avoids p.

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Definition (Elek)

X, Y zero-dimensional Polish spaces; $\Gamma \curvearrowright X$, Y continuously.

We say that X is weakly contained in Y, in symbols $X \preccurlyeq Y$, if for any k, a finite subset $D \subset \Gamma$, and a continuous k-coloring $f \colon X \to k$, there is a continuous k-coloring $g \colon Y \to k$ such that

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Inspired by analogous definitions for measure-preserving actions (KECHRIS) and unitary representations.

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The proof uses the continuous LLL to build a continuous equivariant map $\pi \colon X \to 2^\Gamma$ whose image is "close" to $\mathrm{Free}(2^\Gamma)$.

Combinatorial consequences

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If $\Gamma \curvearrowright X$ is a free continuous action on a nonempty zero-dimensional Polish space X, then $\mathrm{Free}(2^\Gamma) \preccurlyeq X$.

Corollary

Let $\mathscr{P}$ be a finite set of k-patterns. TFAE:

- There is a continuous k-coloring of $\mathrm{Free}(2^\Gamma)$ avoiding all patterns in $\mathscr{P}$.
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For example, $\chi_c(\mathrm{Free}(2^\Gamma)) \geqslant \chi_c(X)$ for all free continuous actions $\Gamma \curvearrowright X$ on zero-dim. Polish spaces.

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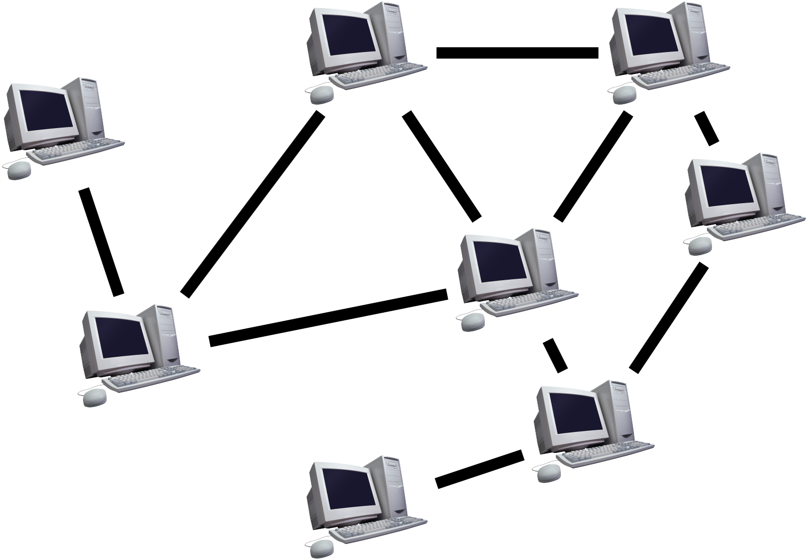
For example, $\chi_c(\mathrm{Free}(2^\Gamma)) \geqslant \chi_c(X)$ for all free continuous actions $\Gamma \curvearrowright X$ on zero-dim. Polish spaces.

Yields a purely combinatorial characterization of finite sets $\mathscr{P}$ of patterns that can be avoided by continuous colorings of $\mathrm{Free}(2^\Gamma)$.

Distributed algorithms



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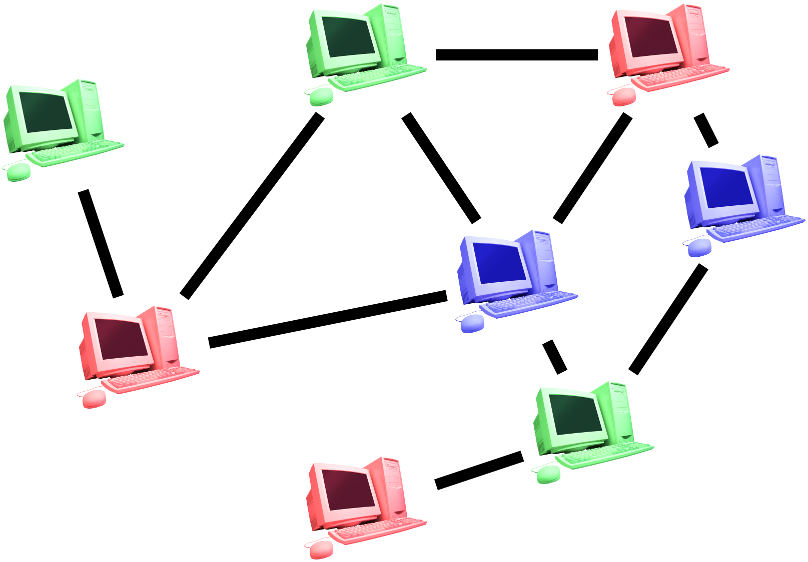

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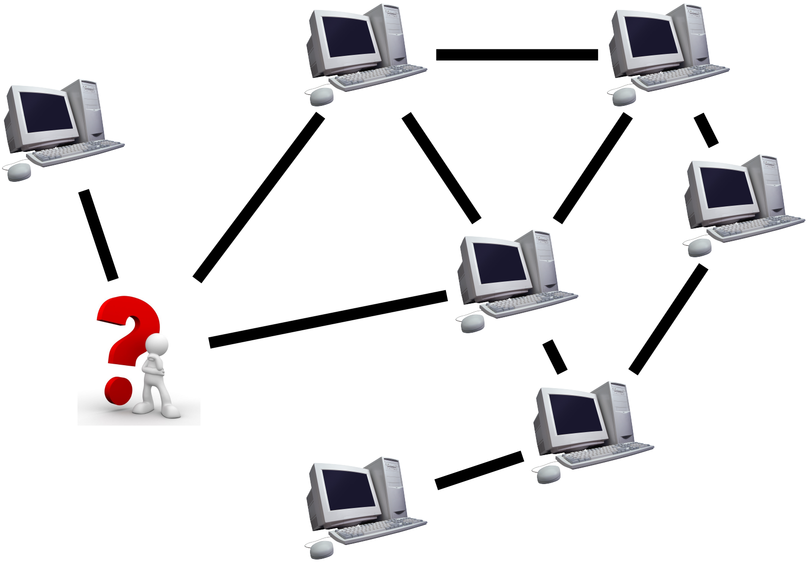

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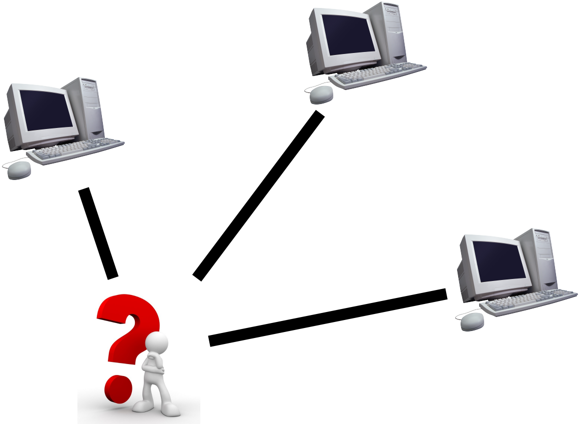

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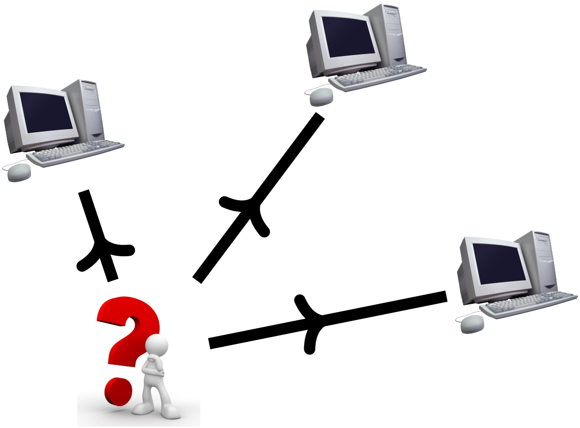

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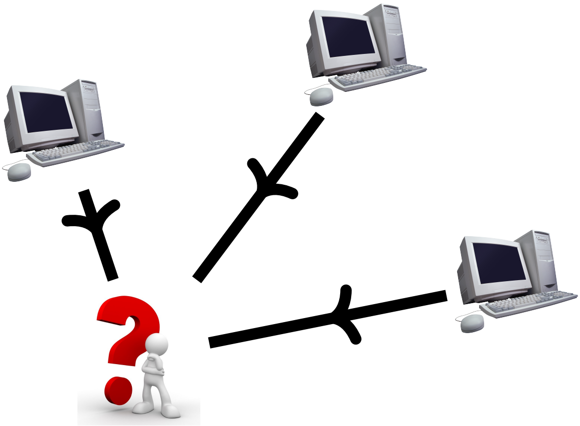

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- Complexity is measured by the number of rounds.
- Each vertex is assigned a unique identifier from $\{1, \ldots, n\}$.
- Every vertex executes the same algorithm, which must always output a correct solution. (The algorithm is deterministic.)

Continuous colorings and deterministic algorithms

Theorem (A.B./Grebík–Jackson–Rozhoň–Seward–Vidnyánszky)

Let $S \subset \Gamma$ be a finite subset. Suppose $\mathscr{P}$ is a finite set of k-patterns such that the domain of every $p \in \mathscr{P}$ is a connected subset of the Cayley graph $\mathrm{Cay}(\Gamma, S)$.

The following statements are equivalent:

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The bound $o(\log n)$ can be improved to $O(\log^* n)$.

Compare and contrast

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GAO–JACKSON–KROHNE–SEWARD: No algorithm can decide, given a finite set $\mathscr{P}$ of k-patterns, whether $\mathrm{Free}(2^\Gamma)$ admits a continuous k-coloring avoiding all patterns in $\mathscr{P}$.

BRANDT et al.: No algorithm can decide, given a finite set $\mathscr{P}$ of k-patterns, whether there is a $o(\log n)$-round LOCAL algorithm that find a k-colorings of $n \times n$ grid graphs avoiding all patterns in $\mathscr{P}$.

Dream

Are there similar combinatorial characterizations for other descriptive regularity notions (Borel, measurable, etc.)?

Thank you!